

The isomorphism problem for multiparameter quantized Weyl algebras^{*†}

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Abstract

In this note we solve the isomorphism problem for the multiparameter quantized Weyl algebras, in the case when none of the deformation parameters q_i is a root of unity, over an arbitrary field.

1 Introduction

The *isomorphism problem* for a family of algebraic structures given by generators and relations, is the problem of determining whether or not two members in the family are isomorphic. This is a very hard problem in general; for the family of finitely presented groups, the problem is known to be undecidable [1], [23].

Quantum algebras are various noncommutative associative algebras arising in the theory of quantum groups. They have many interesting properties from the point of view of ring theory (see e.g. [27], [18], [6], and references therein). When it comes to isomorphisms, the general rule is that quantum algebras are very rigid in the sense of having few automorphisms [12], [3], [28], [29]. This is in stark contrast with the commutative counterparts, where determining automorphism groups can be a wild problem [26].

In this paper we consider a family of quantum algebras, called *multiparameter quantized Weyl algebras*, introduced in [20]. They have been studied in many papers and are related to q -difference operators [17], [20], multiparameter quantum groups [20], twisted generalized Weyl algebras [21], [16], [9] and iterated skew polynomial rings [15], [7]. Some ring theoretic aspects that have been studied for these algebra include global and Krull dimension [11], [8], prime and primitive spectra [25], [15], [13], localizations and division rings of fractions [2], [17]. The automorphism group of a multiparameter quantized Weyl algebra was determined in [25] for generic parameters. The isomorphism problem for

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quantized Weyl algebras $A_1^q(\mathbb{K})$ of degree 1, with arbitrary parameters q , was solved in [10].

Isomorphism problems have been studied for certain classes of *generalized Weyl algebras* [4], [24], [5]. Multiparameter quantized Weyl algebras are not however examples of generalized Weyl algebras, but rather of the more general class of twisted generalized Weyl algebras [21] for which no general results exist about isomorphisms or automorphisms. In [2] some necessary conditions are given for two multiparameter quantized Weyl algebras to be isomorphic.

In the present paper we solve the isomorphism problem for multiparameter quantized Weyl algebras by determining necessary and sufficient conditions for two such algebras to be isomorphic, in the case when certain parameters q_i are not roots of unity. The tools we use are Jordan's simple localization [17] and Rigal's methods [25] for determining automorphisms.

Notation

Throughout, we work over an arbitrary base field \mathbb{K} . For any integers a and b , let $\llbracket a, b \rrbracket$ denote the set $\{x \in \mathbb{Z} \mid a \leq x \leq b\}$.

2 Multiparameter quantized Weyl algebras

Definition 2.1. [20] Let n be a positive integer, $Q = (q_1, \dots, q_n)$ an n -tuple in $(\mathbb{K}^*)^n$, and $\Gamma = (\gamma_{ij})$ a multiplicatively skew-symmetric $n \times n$ matrix with entries from \mathbb{K}^* . The *multiparameter quantized Weyl algebra* $A_n^{Q, \Gamma}(\mathbb{K})$ is defined as the unital associative \mathbb{K} -algebra with generators

$$x_1, y_1, x_2, y_2, \dots, x_n, y_n$$

subject to defining relations

$$y_i y_j = \gamma_{ij} y_j y_i, \quad \forall i, j \in \llbracket 1, n \rrbracket \quad (2.1)$$

$$x_i x_j = q_i \gamma_{ij} x_j x_i, \quad i < j \quad (2.2)$$

$$x_i y_j = \gamma_{ji} y_j x_i, \quad i < j \quad (2.3)$$

$$x_i y_j = q_j \gamma_{ji} y_j x_i, \quad i > j \quad (2.4)$$

$$x_j y_j - q_j y_j x_j = 1 + \sum_{k=1}^{j-1} (q_k - 1) y_k x_k, \quad \forall j \in \llbracket 1, n \rrbracket. \quad (2.5)$$

Frequently used are the elements

$$z_i := [x_i, y_i] = 1 + \sum_{k=1}^i (q_k - 1) y_k x_k, \quad \forall i \in \llbracket 1, n \rrbracket. \quad (2.6)$$

They are normal elements (e.g., [17, §2.8]) and satisfy the following relations ($z_0 = 1$ by convention):

$$x_i y_i - q_i y_i x_i = z_{i-1}, \quad \forall i \in \llbracket 1, n \rrbracket. \quad (2.7)$$

As is well known, $A_n^{Q,\Gamma}(\mathbb{K})$ is an iterated skew polynomial algebra of the form

$$A_n^{Q,\Gamma}(\mathbb{K}) = \mathbb{K}[y_1][x_1; \tau_2, \delta_2][y_2; \tau_3][x_2; \tau_4, \delta_4] \cdots [y_n; \tau_{2n-1}][x_n; \tau_{2n}, \delta_{2n}]$$

(e.g., [17, SS2.1, 2.8]). Consequently, $A_n^{Q,\Gamma}(\mathbb{K})$ is a noetherian domain. The Gelfand-Kirillov dimension of this algebra is $2n$ [14, Proposition 3.4].

3 Height one prime ideals

We let $\langle Q, \Gamma \rangle$ denote the subgroup of \mathbb{K}^* generated by the set

$$\{q_1, \dots, q_n\} \cup \{\gamma_{ij} \mid i, j \in \llbracket 1, n \rrbracket, i < j\}.$$

We say that a multiparameter quantized Weyl algebra $A_n^{Q,\Gamma}(\mathbb{K})$ is *generic* if $\langle Q, \Gamma \rangle$ is free abelian of (the maximal possible) rank $n(n+1)/2$.

Rigal proved in [25, Proposition 3.2.3] that if \mathbb{K} is algebraically closed of characteristic zero and $A_n^{Q,\Gamma}(\mathbb{K})$ is generic, then the height one prime ideals of $A_n^{Q,\Gamma}(\mathbb{K})$ are the principal ideals generated by the z_i . Using results of Jordan [17], we can improve this result by weakening the assumptions on \mathbb{K} , Q and Γ , as follows

Theorem 3.1. *Let $A_n^{Q,\Gamma}(\mathbb{K})$ be a multiparameter quantized Weyl algebra. Assume that for each $i \in \llbracket 1, n \rrbracket$, q_i is not a root of unity. Then the set of height one prime ideals of $A_n^{Q,\Gamma}(\mathbb{K})$ equals*

$$\{(z_1), (z_2), \dots, (z_n)\}.$$

Proof. For each $i \in \llbracket 1, n \rrbracket$, the subalgebra of $A := A_n^{Q,\Gamma}(\mathbb{K})$ generated by the elements $x_1, y_1, \dots, x_i, y_i$ is a multiparameter quantized Weyl algebra of the form $A_i := A_i^{Q^i, \Gamma^i}(\mathbb{K})$, where Q^i and Γ^i denote the restrictions of Q and Γ to $\llbracket 1, i \rrbracket$ and $\llbracket 1, i \rrbracket^2$, respectively. It follows from [17, Proposition 2.7] that z_i generates a completely prime ideal of A_i . In A , we have $(z_i) = z_i A$ by the normality of z_i , from which it follows that $A/(z_i)$ is an iterated skew polynomial ring over $A_i/z_i A_i$, and thus is a domain. Consequently, (z_i) is a completely prime ideal of A .

Next, let Z be the multiplicative submonoid of A generated by the normal elements z_1, z_2, \dots, z_n . Jordan proved in [17, Theorem 3.2] that the localization $A[Z^{-1}]$ is a simple ring. Consequently, any nonzero prime ideal P of A must meet Z , so $z_1^{k_1} z_2^{k_2} \cdots z_n^{k_n} \in P$ for some nonnegative integers k_i , not all equal to zero. Since P is prime and z_i is a normal element in A for every i , it follows that $z_j \in P$ for some $j \in \llbracket 1, n \rrbracket$ and thus $(z_j) \subseteq P$. In particular, if P has height one, then $P = (z_j)$.

Finally, since $(z_j) \not\subseteq (z_i)$ for any $i \neq j$, we conclude that no (z_i) can properly contain a nonzero prime ideal, that is, (z_i) has height one. (This also follows from the Noncommutative Principal Ideal Theorem [22, Theorem 4.1.11].) \square

4 Automorphisms

The automorphisms of $A_n^{Q,\Gamma}(\mathbb{K})$ were determined by Rigal [25, Théorème 4.2.5] under the assumptions that \mathbb{K} is algebraically closed of characteristic zero and $A_n^{Q,\Gamma}(\mathbb{K})$ is generic. These hypotheses were mainly used in determining the height one prime ideals of $A_n^{Q,\Gamma}(\mathbb{K})$. Theorem 3.1 can be used for this purpose, assuming only that the scalars q_i , for $i \in \llbracket 1, n \rrbracket$, are non-roots of unity. The remainder of Rigal's proof carries through as given in [25], yielding the following theorem.

Theorem 4.1. *Let σ be a \mathbb{K} -algebra automorphism of $A_n^{Q,\Gamma}(\mathbb{K})$. If none of q_1, \dots, q_n is a root of unity, there exists $(\mu_1, \dots, \mu_n) \in (\mathbb{K}^*)^n$ such that $\sigma(x_i) = \mu_i x_i$ and $\sigma(y_i) = \mu_i^{-1} y_i$ for all $i \in \llbracket 1, n \rrbracket$.*

We shall also need to upgrade one of the lemmas used in the proof of the above theorem, [25, Lemme 4.2.2], under the same weakened hypotheses.

Lemma 4.2. *Assume that none of q_1, \dots, q_n is a root of unity. Give $A_n^{Q,\Gamma}(\mathbb{K})$ an \mathbb{N} -filtration by specifying total degrees*

$$\begin{aligned} d(x_i) &= d(y_i) = 0, & \forall i \in \llbracket 1, n-1 \rrbracket, \\ d(x_n) &= d(y_n) = 1. \end{aligned}$$

If $a, b \in A_n^{Q,\Gamma}(\mathbb{K}) \setminus \mathbb{K}$ such that $d(ab) = 2$ and $ab \in \text{Span}_{\mathbb{K}}\{z_1, \dots, z_n\}$, then either $(a, b) = (\mu x_n, \nu y_n)$ or $(a, b) = (\mu y_n, \nu x_n)$ for some $\mu, \nu \in \mathbb{K}^$.*

5 Solution to the isomorphism problem

Theorem 5.1. *Let $A_n^{Q,\Gamma}(\mathbb{K})$ and $A_m^{Q',\Gamma'}(\mathbb{K})$ be two multiparameter quantized Weyl algebras, with standard generators x_i, y_i and x'_i, y'_i , respectively.*

- (I) *Assume that q_1, \dots, q_n and q'_1, \dots, q'_m are not roots of unity. Then $A_n^{Q,\Gamma}(\mathbb{K})$ and $A_m^{Q',\Gamma'}(\mathbb{K})$ are isomorphic as \mathbb{K} -algebras if and only if the following two conditions hold:*

- (i) $m = n$;
- (ii) *There exists a sign vector $\varepsilon \in \{1, -1\}^n$ such that for all $i \in \llbracket 1, n \rrbracket$,*

$$q'_i = q_i^{\varepsilon_i}; \tag{5.1}$$

and for all $i, j \in \llbracket 1, n \rrbracket$ with $i < j$,

$$\gamma'_{ij} = \begin{cases} \gamma_{ij}, & \text{if } (\varepsilon_i, \varepsilon_j) = (1, 1), \\ \gamma_{ji}, & \text{if } (\varepsilon_i, \varepsilon_j) = (-1, 1), \\ q_i^{-1} \gamma_{ji}, & \text{if } (\varepsilon_i, \varepsilon_j) = (1, -1), \\ q_i \gamma_{ij}, & \text{if } (\varepsilon_i, \varepsilon_j) = (-1, -1). \end{cases} \tag{5.2}$$

(II) If conditions (i) and (ii) above hold, then for any $\mu \in (\mathbb{K}^*)^n$ and $\varepsilon \in \{\pm 1\}^n$, there exists a unique \mathbb{K} -algebra isomorphism

$$\varphi_{\mu, \varepsilon} : A_n^{Q, \Gamma}(\mathbb{K}) \longrightarrow A_m^{Q', \Gamma'}(\mathbb{K})$$

such that

$$\begin{aligned} \varphi_{\mu, \varepsilon}(x_i) &= \begin{cases} \mu_i x'_i, & \varepsilon_i = 1 \\ \mu_i y'_i, & \varepsilon_i = -1 \end{cases} \\ \varphi_{\mu, \varepsilon}(y_i) &= \begin{cases} \lambda_i \mu_i^{-1} y'_i, & \varepsilon_i = 1 \\ -\lambda_i \mu_i^{-1} x'_i, & \varepsilon_i = -1 \end{cases} \end{aligned}$$

for all $i \in \llbracket 1, n \rrbracket$, where $\lambda = \lambda(\varepsilon) \in (\mathbb{K}^*)^n$ is defined by the recursion relation

$$\lambda_i = q^{(\varepsilon_i - 1)/2} \lambda_{i-1}, \quad \lambda_0 = 1.$$

Finally, $(\mu, \varepsilon) \mapsto \varphi_{\mu, \varepsilon}$ defines a bijection between $(\mathbb{K}^*)^n \times \{\pm 1\}^n$ and the set of isomorphisms $A_n^{Q, \Gamma}(\mathbb{K}) \longrightarrow A_m^{Q', \Gamma'}(\mathbb{K})$.

Proof. Set $A := A_n^{Q, \Gamma}(\mathbb{K})$ and $A' := A_m^{Q', \Gamma'}(\mathbb{K})$, and set $z'_i := [x'_i, y'_i] \in A'$ for $i \in \llbracket 1, m \rrbracket$, as well as $z'_0 = 1 \in A'$. For $i \in \llbracket 1, n \rrbracket$, write A_i for the subalgebra of A generated by $x_1, y_1, \dots, x_i, y_i$, and write A'_j for the analogous subalgebras of A' .

(I): Suppose $\varphi : A \rightarrow A'$ is an isomorphism of \mathbb{K} -algebras. Since the Gelfand-Kirillov dimensions of A and A' are $2n$ and $2m$, it is immediate that $m = n$. Now φ must map the set of height one prime ideals of A bijectively onto the corresponding set for A' . In view of Theorem 3.1, we conclude that there exist $\lambda \in (\mathbb{K}^*)^n$ and a permutation $\sigma \in S_n$ such that

$$\varphi(z_i) = \lambda_i z'_{\sigma(i)}, \quad \forall i \in \llbracket 1, n \rrbracket.$$

Give A' the \mathbb{N} -filtration specified in Lemma 4.2. Let $i = \sigma^{-1}(n)$. By definition of z_i we have $z_i - z_{i-1} = (q_i - 1)y_i x_i$. Applying φ to both sides gives

$$\lambda z'_n - \varphi(z_{i-1}) = (q_i - 1)\varphi(y_i)\varphi(x_i),$$

which implies

$$\lambda(q'_n - 1)y'_n x'_n + c = (q_i - 1)\varphi(y_i)\varphi(x_i)$$

where $d(c) = 0$, and so $d(\varphi(y_i)\varphi(x_i)) = 2$. Now we use Lemma 4.2. We obtain that for $i = \sigma^{-1}(n)$,

$$(\varphi(x_i), \varphi(y_i)) \in \{(\mu x'_n, \nu y'_n), (\mu y'_n, \nu x'_n) \mid \mu, \nu \in \mathbb{K}^*\}. \quad (5.3)$$

Suppose $i < n$. Then we can apply the same argument as above to the relation $z_{i+1} - z_i = (q_{i+1} - 1)y_{i+1}x_{i+1}$ and obtain

$$(\varphi(x_{i+1}), \varphi(y_{i+1})) \in \{(\mu x'_n, \nu y'_n), (\mu y'_n, \nu x'_n) \mid \mu, \nu \in \mathbb{K}^*\}. \quad (5.4)$$

But then (5.3) and (5.4) imply that x'_n has two different pre-images under φ , which is obviously impossible. Therefore $i = n$. So $\varphi(z_n) = \lambda_n z'_n$ and

$$(\varphi(x_n), \varphi(y_n)) \in \{(\mu x'_n, \nu y'_n), (\mu y'_n, \nu x'_n) \mid \mu, \nu \in \mathbb{K}^*\}. \quad (5.5)$$

In particular, for $j < n$ we have $d(\varphi(y_j)\varphi(x_j)) = d(\varphi(z_j) - \varphi(z_{j-1})) = 0$ which shows that $\varphi(x_j), \varphi(y_j) \in A'_{n-1}$. So $\varphi(A_{n-1}) \subseteq A'_{n-1}$. Hence, applying the argument leading to (5.5) to $\varphi|_{A_j}$ for $j = n-1, n-2, \dots, 1$ we obtain

$$\begin{aligned} \varphi(z_i) &= \lambda_i z'_i, \quad \forall i \in \llbracket 1, n \rrbracket, \\ (\varphi(x_i), \varphi(y_i)) &\in \{(\mu_i x'_i, \nu_i y'_i), (\mu_i y'_i, \nu_i x'_i) \mid \mu_i, \nu_i \in \mathbb{K}^*\}, \quad \forall i \in \llbracket 1, n \rrbracket. \end{aligned}$$

We define $\varepsilon \in \{1, -1\}^n$ by

$$\varepsilon_i = \begin{cases} 1, & \text{if } (\varphi(x_i), \varphi(y_i)) = (\mu_i x'_i, \nu_i y'_i) \text{ for some } \mu_i, \nu_i \in \mathbb{K}^*, \\ -1, & \text{if } (\varphi(x_i), \varphi(y_i)) = (\mu_i y'_i, \nu_i x'_i) \text{ for some } \mu_i, \nu_i \in \mathbb{K}^*. \end{cases}$$

Next we prove that (5.1) holds. Applying φ to the relation $x_i y_i - q_i y_i x_i = z_{i-1}$ we obtain

$$\begin{aligned} \mu_i \nu_i (x'_i y'_i - q_i y'_i x'_i) &= \lambda_{i-1} z'_{i-1}, & \varepsilon_i &= 1, \\ \mu_i \nu_i (y'_i x'_i - q_i x'_i y'_i) &= \lambda_{i-1} z'_{i-1}, & \varepsilon_i &= -1. \end{aligned}$$

Using $x'_i y'_i - q'_i y'_i x'_i = z'_{i-1}$, we get

$$\begin{aligned} \mu_i \nu_i (q'_i - q_i) y'_i x'_i &\in A'_{i-1}, & \varepsilon_i &= 1, \\ \mu_i \nu_i ((q'_i)^{-1} - q_i) x'_i y'_i &\in A'_{i-1}, & \varepsilon_i &= -1. \end{aligned}$$

Statement (5.1) follows immediately.

Let $i, j \in \llbracket 1, n \rrbracket$. Assume that $i < j$. We apply φ to relation (2.1) to obtain

$$\begin{aligned} 0 &= (\nu_i \nu_j)^{-1} (\varphi(y_i) \varphi(y_j) - \gamma_{ij} \varphi(y_j) \varphi(y_i)) \\ &= \begin{cases} y'_i y'_j - \gamma_{ij} y'_j y'_i = (\gamma'_{ij} - \gamma_{ij}) y'_j y'_i & (\varepsilon_i, \varepsilon_j) = (1, 1) \\ x'_i y'_j - \gamma_{ij} y'_j x'_i = (\gamma'_{ji} - \gamma_{ij}) y'_j x'_i & (\varepsilon_i, \varepsilon_j) = (-1, 1) \\ y'_i x'_j - \gamma_{ij} x'_j y'_i = (1 - \gamma_{ij} q'_i \gamma'_{ij}) y'_i x'_j & (\varepsilon_i, \varepsilon_j) = (1, -1) \\ x'_i x'_j - \gamma_{ij} x'_j x'_i = (q'_i \gamma'_{ij} - \gamma_{ij}) x'_j x'_i & (\varepsilon_i, \varepsilon_j) = (-1, -1). \end{cases} \end{aligned}$$

Combining this with (5.1), we obtain (5.2). This proves the necessity of conditions (i), (ii).

Conversely, assume that (i), (ii) hold. Define elements

$$x'_i = \begin{cases} x'_i, & \varepsilon_i = 1 \\ y'_i, & \varepsilon_i = -1 \end{cases} \quad \text{and} \quad y'_i = \begin{cases} \lambda_i y'_i, & \varepsilon_i = 1 \\ -\lambda_i x'_i, & \varepsilon_i = -1 \end{cases}$$

in A for $i \in \llbracket 1, n \rrbracket$, where $\lambda = \lambda(\varepsilon) \in (\mathbb{K}^*)^n$ is defined by the recursion relation

$$\lambda_i = q_i^{(\varepsilon_i - 1)/2} \lambda_{i-1}, \quad \lambda_0 = 1.$$

It is straightforward to verify that these elements satisfy the relations (2.1)-(2.4). As for (2.5), note that this is equivalent to

$$\begin{aligned} x_1^* y_1^* - q_1 y_1^* x_1^* &= 1, \\ (x_j^* y_j^* - q_j y_j^* x_j^*) - (x_{j-1}^* y_{j-1}^* - q_{j-1} y_{j-1}^* x_{j-1}^*) &= (q_{j-1} - 1) y_{j-1}^* x_{j-1}^*, \\ \forall j \in \llbracket 2, n \rrbracket, \end{aligned} \quad (5.6)$$

and that (5.7) is equivalent to $x_j^* y_j^* - q_j y_j^* x_j^* = [x_{j-1}^*, y_{j-1}^*]$. For both choices of ε_{j-1} , the commutator $[x_{j-1}^*, y_{j-1}^*]$ equals $\lambda_{j-1} [x'_{j-1}, y'_{j-1}]$. Consequently, (5.6) and (5.7) will both follow from

$$x_j^* y_j^* - q_j y_j^* x_j^* = \lambda_{j-1} z'_{j-1}, \quad \forall j \in \llbracket 1, n \rrbracket. \quad (5.8)$$

It is straightforward to verify (5.8).

Since the elements x_i^*, y_i^* satisfy the relations (2.1)-(2.5), there is a unique \mathbb{K} -algebra homomorphism $\varphi : A \rightarrow A'$ such that

$$\varphi(x_i) = x_i^* \quad \text{and} \quad \varphi(y_i) = y_i^*, \quad \forall i \in \llbracket 1, n \rrbracket.$$

It is clearly surjective, since its image contains x'_i and y'_i for all $i \in \llbracket 1, n \rrbracket$. Injectivity follows because A and A' are domains with the same finite Gelfand-Kirillov dimension (see [19, Proposition 3.15]).

(II): This follows from the proof of part (I) and Theorem 4.1. \square

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